REMARKS ON SOME NONLINEAR EVOLUTION PROBLEMS ARISING IN BINGHAM FLOWS

BY

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ABSTRACT

A Bingham flow is described by a so-called variational inequality of evolution type which contains the Navier Stokes equations as a particular case. These variational inequalities were introduced and studied by Duvaut and the author. We recall here a number of known results for these "Bingham inequalities" and initiate the study of the behaviour of the solution when the "viscosity" tends to zero.

1. Introduction

Bingham fluids are non-Newtonian fluids whose law of behaviour is given by Duvaut and Lions [6, Chap. 6]. The law depends on two (main) parameters: the viscosity μ (> 0) and the plasticity yield $g > 0$; when $g = 0$ one recovers the usual viscous incompressible fluids, leading to Navier Stokes equations. When g is >0, the speed u of the flow can be characterized by a *variational inequality of evolution*, which has been introduced and studied by Duvaut and Lions [6], [7]. We recall in Section 2 below the main known results for this variational inequality (which contains the Navier Stokes equations for $g = 0$).

We consider in Section 3 the problem of the behaviour of the solution (which is known to exist and to be unique if the space dimension equals 2) *when the viscosity μ tends to zero.* We give a result which is a *partial* extension to the case "$g > 0$" of a result of Yudovich [40] known for the case "$g = 0$".

In Section 4 we consider, following Duvaut and Lions [8], a case when *the viscosity μ depends on the temperature.*

We give in Section 5 a number of remarks on related questions and open problems.

2. Variational Bingham inequalities

Let Ω be a bounded open set of $\mathbb{R}^n$, $n = 2$ or 3[†], with smooth boundary Γ. Let $u = u(x,t)$ denote the speed of the flow. We introduce the following notations:

$$D_{ij}(v) = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

$$D_{II}(v) = \tfrac{1}{2}\, D_{ij}(v) D_{ij}(v),^{\dagger\dagger}$$

$$a(v,w) = 2 \int_\Omega D_{ij}(v) D_{ij}(w)\, dx,$$

$$b(u,v,w) = \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j\, dx,$$

$$j(v) = 2 \int_\Omega D_{II}(v)^{\frac{1}{2}} dx.$$

Denote by $\mathscr{V}$ the space

$$\mathscr{V} = \{\phi \mid \phi \in (\mathscr{D}(\Omega))^n,\ \mathrm{Div}\, \phi = 0\}$$

where $\mathscr{D}(\Omega) = C_0^\infty(\Omega) =$ space of (real valued) C^∞ functions defined in Ω with compact support.

Let $H^s(\Omega)$ be the usual Sobolev space of order s ($\geqq 0$), s being an integer or not. We define next

$$V_s = \text{closure of } \mathscr{V} \text{ in } (H^s(\Omega))^n. \tag{2.1}$$

We set, in particular,

$$V_1 = V, \qquad V_0 = H. \tag{2.2}$$

We denote by $\| \ \|$ (resp. $| \ |$, resp. $\| \ \|_s$) the norm in V (resp. in H, resp. in V_s), and by $(\ , \ )$ the scalar product in H

The problem of Bingham flows can now be stated in its *strong form* in the following manner: We are looking for a function

$$t \to u(t) \text{ from } [0,T] \to V \text{ such that}$$

$$\begin{aligned}\left(\frac{\partial u(t)}{\partial t}, v - u(t)\right) + \mu a(u(t), v - u(t)) + b(u(t), u(t), v - u(t))\\ + gj(v) - gj(u(t)) \geqq (f(t), v - u(t)), \ \forall v \in V\end{aligned} \tag{2.3}$$

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† We note that one has an existence theorem with arbitrary n; cf. Duvaut and Lions [6].

†† We follow the standard summation convention.

and

(2.4) $$u(0) = u_0,\ u_0 \text{ given in } H.$$

In (2.3), f is given satisfying

(2.5) $$f \in L^2(0, T;\ V').^{\dagger}$$

Inequalities (2.3) are called *variational Bingham inequalities.*

REMARK 2.1. In (2.3), the term $b(u(t),\ u(t),\ u(t)) = 0$, but it is kept for the symmetry of the formula.

REMARK 2.2. When $g = 0$, (2.3) reduces to the Navier Stokes equations in the usual variational form (cf. J. Leray [21], [22]; E. Hopf [14]) namely

(2.6) $$\left(\frac{\partial u(t)}{\partial t}, v\right) + \mu\, a(u(t)),v) + b(u(t),u(t),v) = (f(t),v), \qquad \forall v \in V.$$

The problem (2.3) (2.4) is solved in a satisfying manner when $n = 2$. One has

THEOREM 2.1. *We suppose that $n = 2$ and that f and u_0 are given satisfying* (2.5) (2.4). *Then there exists a unique function u which satisfies*

(2.7) $$u \in L^2(0, T\ ;\ V),$$

(2.8) $$\frac{\partial u}{\partial t} \in L^2(0.\ T;\ V')$$

and which satisfies (2.3).

REMARK 2.3. It follows from (2.7) (2.8) that the function u a.e. equals a continuous function $t \to u(t)$ from $[0, T] \to H$.

REMARK 2.4. When $g = 0$, Theorem 2.1 gives a result of Lions and Prodi [27]. One can also give *regularity theorems* in t which give, as particular cases when $g = 0$, results of Ladyzenskaya [18].

REMARK 2.5. The solution u given in Theorem 2.1 depends on μ and g:

$$u = u_{\mu,g}.$$

For fixed $\mu > 0$, $u_{\mu,g}$ converges (in the topology which corresponds to (2.7) (2.8)) to u_μ = usual solution of Navier Stokes equations when $g \to 0$. The case when

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† We identify H to its dual; V_s' denotes the dual of V_s; If $s > 1$ we have then $V_s \subset V \subset H \subset V' \subset V_s'$.

$g \to +\infty$ is also of interest; we shall give some indications on this case in Remark 5 hereafter.

The problem of the behaviour of $u_{\mu,g}$ when $\mu \to 0$ $(g > 0)$ is considered in Section 3 below.

We now return to the problem (2.3), when $n = 3$. We need in this case the notion of *weak solution* of the Bingham inequalities.
Introduce the space W defined by

$$W = \left\{ v \,\middle|\, v \in L^2(0,T;\, V_{3/2}),\ \frac{\partial v}{\partial t} \in L^2(0,T;\, H), v(o) = 0 \right\} \tag{2.9}$$

and to simplify the exposition assume that

$$u_0 = 0. \tag{2.10}$$

We remark that if u satisfies (2.3) (2.4), then, $\forall\, v \in W$ we have

$$\int_0^T \left[\left(\frac{\partial v}{\partial t}, v - u \right) + \mu a(u, v - u) + b(u,u,v-u) + gj(v) - gj(u) - (f, v-u) \right] dt \geqq 0. \tag{2.11}$$

We eliminate in (2.11) the term $b(u,u,u)$ (which is zero for strong solutions) and we shall say that u is a *weak solution* if

$$\left\{ \int_0^T \left[\left(\frac{\partial v}{\partial t}, v - u \right) + \mu a(u, v - u) + b(u,u,v) + gj(v) - gj(u) - (f, v-u) \right] dt \geqq 0 \qquad \forall\, v \in W. \right. \tag{2.12}$$

We have

THEOREM 2.2. *Assume that $n = 3$, f is given satisfying* (2.5) *and* (2.10) *holds true. Then there exists a function u which satisfies*

$$u \in L^2(0,T;\, V) \cap L^\infty(0,T;\, H), \tag{2.13}$$

$$\frac{\partial u}{\partial t} \in L^2\,(0,T;\, V'_{3/2}) \tag{2.14}$$

and which satisfies (2.12).

REMARK 2.6. The *uniqueness* in Theorem 2.2 is an *open problem*; actually the situation is completely analogous to the one of the usual Navier Stokes equations (although the methods of proof are quite different).

REMARK 2.7. We refer to Duvaut and Lions [6] for the *stationary* case.

3. A remark on the problem when the viscosity tends to zero

We consider the case $n = 2$ and we suppose that Ω is simply connected. Coordinates are denoted by $\{x_1, x_2\}$ or $\{x, y\}$.
Introduce the stream function ψ which is uniquely defined by

$$\frac{\partial \psi}{\partial x} = \psi_x = -u_2, \quad \frac{\partial \psi}{\partial y} = \psi_y = u_1 \tag{3.1}$$

and

$$\psi = 0 \text{ on } \Gamma. \tag{3.2}$$

If, as in Section 2, $u_1 = u_2 = 0$ on Γ, then

$$\frac{\partial \psi}{\partial n} = 0 \text{ on } \Gamma. \tag{3.3}$$

In what follows, we shall consider *another boundary condition* rather than (3.3), namely,

$$\Delta \psi = 0 \text{ on } \Gamma. \tag{3.4}$$

REMARK 3.1. A similar change is made, in the case $g = 0$, by Yudovich [40]. We now transform the variational inequality (2.3) in terms of ψ. To every test function v we associate ϕ uniquely defined by

$$\phi_x = -v_2, \quad \phi_y = v_1, \quad \phi = 0 \quad \text{on } \Gamma. \tag{3.5}$$

We check the following formulas:

$$D_{II}(v) = \phi_{xy}^2 + \tfrac{1}{4}(\phi_{xx} - \phi_{yy})^2, \tag{3.6}$$

$$a(u, v) = \mathscr{A}(\psi, \phi) = (\Delta\psi, \Delta\phi)^{\dagger} \tag{3.7}$$

$$\begin{cases} j(v) = \mathscr{J}(\phi) = 2\displaystyle\int_\Omega M(\phi)^{\frac{1}{2}} dx\, dy, \\ M(\phi) = \phi_{xy}^2 + \tfrac{1}{4}(\phi_{xx} - \phi_{yy})^2 \end{cases} \tag{3.8}$$

$$\begin{aligned} b(u,u,v) &= \beta(\psi, \psi, \phi) \\ &= \int_\Omega [\psi_y \Delta\psi \phi_x - \psi_x \Delta\psi \phi_y]\, dx\, dy, \end{aligned} \tag{3.9}$$

† We set $(f, g) = \int_\Omega fg dx\, dy$.

$$(u,v) = a_0(\psi,\phi) = \int_\Omega [\psi_x\phi_x + \psi_y\phi_y]\,dx\,dy, \tag{3.10}$$

$$(f,v) = (F,\phi),\ F = \frac{\partial}{\partial x}f_2 - \frac{\partial}{\partial y}f_1. \tag{3.11}$$

Variational inequality (2.3) becomes now

$$\begin{cases} a_0\left(\dfrac{\partial\psi}{\partial t}, \phi - \psi\right) + \mu\mathscr{A}(\psi,\phi-\psi) + \beta(\psi,\psi,\phi-\psi) \\ \qquad + g\mathscr{J}(\phi) - g\mathscr{J}(\psi) \geqq (F,\phi-\psi) \qquad \forall\phi \\ \text{such that } \phi = 0,\ \Delta\phi = 0 \text{ on } \Gamma. \end{cases} \tag{3.12}$$

The *new* problem we consider is now: *to find a function* ψ *satisfying the boundary conditions* (3.2) (3.4), *the variational inequality* (3.12) *and the initial conditions*:

$$\psi\,\big|_{t=0} = 0.^{\dagger} \tag{3.13}$$

REMARK 3.2. Since we have replaced (3.3) by (3.4) we *cannot* apply Theorem 3.1 to obtain the existence and uniqueness of a solution ψ. We shall prove:

THEOREM 3.1. *Assume that*

$$\Omega =]o,d[^2 \tag{3.14}$$

and that f_1, f_2 *are given satisfying*

$$f_1,\ f_2,\ F \in L^2(Q), \quad Q = \Omega x]0,T[. \tag{3.15}$$

For $\mu > 0$ *fixed, there exists a unique function* $\psi = \psi^\mu$ *which satisfies*

$$\psi \in L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega)),^{\dagger\dagger} \tag{3.16}$$

$$\Delta\psi \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)), \tag{3.17}$$

$$\frac{\partial}{\partial t}(-\Delta\psi) \in L^2(0,T;(H^2(\Omega)\cap H_0^1(\Omega))'), \tag{3.18}$$

and (3.12) (3.13).

Moreover, as $\mu \to 0$, *there exists a constant* C *which does not depend on* μ, *such that*

$$\begin{aligned} &\|\psi^\mu\|_{L^\infty(0,T;H_0^1(\Omega))} + \|\Delta\psi^\mu\|_{L^\infty(0,T;L^2(\Omega))} \\ &\qquad + \left\|\frac{\partial}{\partial t}(-\Delta\psi^\mu)\right\|_{L^2(0,T;(H^2(\Omega)\cap H_0^1(\Omega))')} \leqq C. \end{aligned} \tag{3.19}$$

† We take "0" to simplify the exposition.

†† $H_0^1(\Omega)$ denotes the closure of $\mathscr{D}(\Omega)$ in $H^1(\Omega)$

We can then let μ tend to zero, to obtain

THEOREM 3.2. *Under the hypothesis of Theorem* 3.1, *we can extract a subsequence, still denoted by* ψ^μ, *of solutions such that when* $\mu \to 0$ *one has*

$$\psi^\mu \to \psi \text{ in } L^\infty(0,T;H_0^1(\Omega)) \text{ weak-star,} \tag{3.20}$$

$$\Delta\psi^\mu \to \Delta\psi \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak-star,} \tag{3.21}$$

$$\frac{\partial}{\partial t}(-\Delta\psi^\mu) \to \frac{\partial}{\partial t}(-\Delta\psi) \text{ in } L^2(0,T;(H^2(\Omega)\cap H_0^2(\Omega))') \text{ weakly} \tag{3.22}$$

where ψ *is a solution of the variational inequality*

$$\begin{cases} a_0\left(\dfrac{\partial\psi}{\partial t}, \phi - \psi\right) + \beta(\psi,\psi,\phi) + g\mathscr{J}(\phi) - g\mathscr{J}(\psi) \geqq (F, \phi - \psi) \\ \forall\, \phi \text{ such that } \phi = 0 \text{ on } \Gamma \end{cases} \tag{3.23}$$

and satisfies

$$\psi\,|_{t=0} = 0. \tag{3.24}$$

PROOF OF THEOREM 3.1. *Existence.*

We firstly *regularize* $\mathscr{J}(\phi)$; we define, for $\varepsilon > 0$

$$\mathscr{J}_\varepsilon(\phi) = 2\int_\Omega M(\phi)^{(1+\varepsilon)/2}\,dx\,dy \tag{3.25}$$

where $M(\phi)$ is defined in (3.8). Then the functional $\phi \to \mathscr{J}_\varepsilon(\phi)$ is differentiable and we have

$$\begin{aligned}&(\mathscr{J}'_\varepsilon(\psi),\phi)\\ &= 2(1+\varepsilon)\int_\Omega M(\psi)^{(\varepsilon-1)/2}(\psi_{xy}\phi_{xy} + \tfrac{1}{4}(\psi_{xx} - \psi_{yy})(\phi_{xx} - \phi_{yy}))\,dxdy.\end{aligned} \tag{3.26}$$

We then "approximate" (3.12) by the nonlinear P.D.E.

$$a_0(\psi'_\varepsilon, \phi) + \mu\mathscr{A}(\psi_\varepsilon, \phi) + \beta(\psi_\varepsilon, \psi_\varepsilon, \phi) + g(\mathscr{J}'_\varepsilon(\psi_\varepsilon), \phi) = (F, \phi) \tag{3.27}$$

and

$$\psi_\varepsilon(o) = 0 \tag{3.28}$$

(we have denoted $\partial\psi_\varepsilon/\partial t$ by ψ'_ε).

The solution of (3.27) is constructed by the Galerkin method with a "special basis" as in Lions [23] (for the case $g = 0$). We consider the eigen functions w_m:

$$-\Delta w_m = \lambda_m w_m, \qquad w_m = 0 \text{ on } \Gamma \tag{3.29}$$

and we apply the Galerkin method with the w_m basis, i.e., we define $\psi_{\varepsilon m} =$ approximate solution of (3.27) (3.28) in the following manner:

$$\begin{cases} \psi_{\varepsilon m}(t) \in [w_1, \cdots, w_m] = \text{space spanned by } w_1, \cdots, w_m; \\ a_0(\psi'_{\varepsilon m}, \phi) + \mu\mathscr{A}(\psi_{\varepsilon m}, \phi) + \beta(\psi_{\varepsilon m}, \psi_{\varepsilon m}, \phi) + g(\mathscr{J}'_\varepsilon(\psi_{\varepsilon m}), \phi) \\ \qquad = (F, \phi) \qquad \forall \phi \in [w_1, \cdots, w_m], \\ \psi_{\varepsilon m}(o) = 0. \end{cases} \tag{3.30}$$

We are now going to obtain estimates on $\psi_{\varepsilon m}$ which do not depend on ε and m and we shall see how these estimates depend on μ.

Estimates (I). The first estimates are straightforward; we replace in (3.30) ϕ by $\psi_{\varepsilon m}$; we observe that $\beta(\phi, \phi, \phi) = 0$ and that $(\mathscr{J}'_\varepsilon(\phi), \phi) \geqq 0$; we obtain therefore:

$$\tfrac{1}{2} \frac{d}{dt} a_0(\psi_{\varepsilon m}, \psi_{\varepsilon m}) + \sqrt{\mu}\mathscr{A}(\psi_{\varepsilon m}, \psi_{\varepsilon m}) \leqq (F, \psi_{\varepsilon m}). \tag{3.31}$$

Hence we easily deduce that †

$$\|\psi_{\varepsilon m}\|_{L^\infty(0\ T;\, H_0^1(\Omega))} + \sqrt{\mu}\, \|\Delta\psi_{\varepsilon m}\|_{L^2(0,T;\, L^2(\Omega))} \leqq C. \, ^{\dagger\dagger} \tag{3.32}$$

Estimates (II). Since w_m satisfies (3.29), it follows from (3.30) that

$$a_0(\psi'_{\varepsilon m}, -\Delta w_j) + \mu\mathscr{A}(\psi_{\varepsilon m}, -\Delta w_j) + \beta(\psi_{\varepsilon m}, \psi_{\varepsilon m}, -\Delta w_j) + g(\mathscr{J}'_\varepsilon(\psi_{\varepsilon m}), -\Delta w_j) = (F, -\Delta w_j).$$

Hence we obtain

$$\begin{aligned} &a_0(\psi'_{\varepsilon m}, -\Delta\psi_{\varepsilon m}) + \mu\mathscr{A}(\psi_{\varepsilon m}, -\Delta\psi_{\varepsilon m}) + \beta(\psi_{\varepsilon m}, \psi_{\varepsilon m}, -\Delta\psi_{\varepsilon m}) \\ &\quad + g(\mathscr{J}'_\varepsilon(\psi_{\varepsilon m}), -\Delta\psi_{\varepsilon m}) = (F, -\Delta\psi_{\varepsilon m}). \end{aligned} \tag{3.33}$$

But we remark that

$$\begin{aligned} \beta(\phi, \phi, -\Delta\phi) &= \tfrac{1}{2} \int_\Omega \left[\phi_x \frac{\partial}{\partial y}(\Delta\phi)^2 - \phi_y \frac{\partial}{\partial x}(\Delta\phi)^2\right] dx\, dy \\ &= \tfrac{1}{2} \int_\Gamma [\phi_x \cos(n, y) - \phi_y \cos(n, x)]\, (\Delta\phi)^2 d\Gamma = 0, \end{aligned} \tag{3.34}$$

Since the tangential derivative of ϕ is zero on Γ.

† We obtain simultaneously the existence in $(0,T)$ of a solution of (3.30).

†† The constants C do not depend on ε, m, μ.

Let us consider for a moment

LEMMA 3.1. *For every smooth function ϕ such that $\phi = \Delta\phi = 0$ on Γ we have*

$$(\mathscr{J}'_\varepsilon(\phi), -\Delta\phi) \geqq 0. \tag{3.35}$$

Then (3.33) gives

$$\tfrac{1}{2} \frac{d}{dt} |\Delta\psi_{\varepsilon m}|^2 + \mu a_0(\Delta\psi_{\varepsilon m}, \Delta\psi_{\varepsilon m}) \leqq (F, -\Delta\psi_{\varepsilon m})^\dagger \tag{3.36}$$

Hence we easily obtain[††]

$$\| \psi_{\varepsilon m} \|_{L\infty(0,T,H^2(\Omega))} + \sqrt{\mu} \| \Delta\psi_{\varepsilon m} \|_{L^2(0,T;H_0^1(\Omega))} \leqq C. \tag{3.37}$$

PROOF OF LEMMA 3.1. We have, setting $M(\psi) = M$:

$$\begin{cases} (\mathscr{J}'_\varepsilon(\phi), -\Delta\phi) \\ = (1+\varepsilon)\int_\Omega M^{(\varepsilon-1)/2}\Big[2\phi_{xy}(-\Delta\phi_{xy}) + \tfrac{1}{2}(\phi_{xx}-\phi_{yy})(-\Delta(\phi_{xx}-\phi_{yy}))\Big]dxdy \\ = (1+\varepsilon)\int_\Gamma M^{(\varepsilon-1)/2}\Big[2\phi_{xy}\Big(-\frac{\partial\phi_{xy}}{\partial n}\Big) \\ \quad + \tfrac{1}{2}(\phi_{xx}-\phi_{yy})\Big(-\frac{\partial}{\partial n}(\phi_{xx}-\phi_{yy})\Big)\Big]d\Gamma \\ \quad + (1+\varepsilon)X, \end{cases} \tag{3.38}$$

where

$$\begin{aligned} X = \int_\Omega \Big[& (2M^{(\varepsilon-1)/2}\phi_{xy})_x\phi_{xxy} + (2M^{(\varepsilon-1)/2}\phi_{xy})_y\phi_{xyy} \\ & + \tfrac{1}{2}(M^{(\varepsilon-1)/2}(\phi_{xx}-\phi_{yy}))_x(\phi_{xxx}-\phi_{yyy}) \\ & + \tfrac{1}{2}(M^{(\varepsilon-1)/2}(\phi_{xx}-\phi_{yy}))_y(\phi_{xxy}-\phi_{yyy})\Big]\,dx\,dy. \end{aligned} \tag{3.39}$$

We obtain

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† $|f|^2 = \int_\Omega f^2\,dx\,dy.$

†† We use the fact that $\| \phi \|_{H^2(\Omega)} \leqq c\,|\Delta\phi|$ if $\phi = 0$ on Γ.

$$X = \int_\Omega M^{(\varepsilon-1)/2}\Big[2(\phi_{xxy}^2+\phi_{xyy}^2)+\tfrac12((\phi_{xx}-\phi_{yy})_x)^2+\tfrac12((\phi_{xx}-\phi_{yy})_y)^2\Big]\,dx\,dy$$

$$+\int_\Omega (M^{(\varepsilon-1)/2})_x\Big[2\,\phi_{xy}\phi_{xxy}+\tfrac12(\phi_{xx}-\phi_{yy})(\phi_{xx}-\phi_{yy})_x\Big]\,dx\,dy$$

$$+\int_\Omega (M^{(\varepsilon-1)/2})_y\Big[2\phi_{xy}\phi_{xyy}+\tfrac12(\phi_{xx}-\phi_{yy})(\phi_{xx}-\phi_{yy})_y\Big]\,dx\,dy$$

$$= \int_\Omega M^{(\varepsilon-1)/2}\Big[2(\phi_{xxy}^2+\phi_{xyy}^2)+\tfrac12(\phi_{xx}-\phi_{yy})_x)^2+\tfrac12((\phi_{xx}-\phi_{yy})_y)^2\Big]\,dx\,dy$$

$$+\int_\Omega\Big[(M^{(\varepsilon-1)/2})_xM_x+(M^{(\varepsilon-1)/2})_yM_y\Big]\,dx\,dy$$

$$= \frac{\varepsilon}{2}\int_\Omega M^{(\varepsilon-3)/2}(M_x^2+M_y^2)\,dx\,dy+\int_\Omega M^{(\varepsilon-3)/2}Y\,dxdy$$

where

$$Y = M\Big[2(\phi_{xxy}+\phi_{xyy}^2)+\tfrac12((\phi_{xx}-\phi_{yy})_x)^2+\tfrac12((\phi_{xx}-\phi_{yy})_y{}^2\Big]$$

$$-\ \tfrac12\Big[M_x^2+M_y^2\Big].$$

If we set

$$\phi_{xx}-\phi_{yy}=\sigma$$

we check that

$$Y=\tfrac12(\phi_{xy}\sigma_x-\phi_{xxy}\sigma)^2+\tfrac12(\phi_{xy}\sigma_y-\phi_{xyy}\sigma)^2$$

hence $Y \geqq 0$, so that $X \geqq 0$ and therefore (3.38) gives

$$(\mathscr{J}_\varepsilon'(\phi),-\Delta\phi)\geqq-(1+\varepsilon)\int_\Gamma M^{(\varepsilon-1)/2}Z\,d\Gamma \tag{3.40}$$

where

$$Z = 2\phi_{xy}\,\frac{\partial}{\partial n}\phi_{xy}+\tfrac12\,\sigma\,\frac{\partial}{\partial n}\sigma$$

$$= \frac{\partial}{\partial n}\Big[\phi_{xy}^2+\tfrac14\sigma^2\Big]$$

$$= \frac{\partial}{\partial n}\Big[\phi_{xy}^2-\phi_{xx}\phi_{yy}+\tfrac14(\Delta\phi)^2\Big].$$

But since $\Delta\phi=0$ on Γ, it follows that $\partial/\partial n\,(\Delta\phi)^2=0$ on Γ so that

$$Z=\frac{\partial}{\partial n}\Big[\phi_{xy}-\phi_{xx}\phi_{yy}\Big]\ \text{on}\ \Gamma.$$

But one easily checks that since $\phi = 0$, $\Delta\phi = 0$ on Γ; then on each part of $\Gamma = \partial\Omega$, $\Omega =]o, d[^2$, one has $z = 0$; hence (3.40) gives (3.35).

Estimate (III). We now show that

$$\left\| \frac{\partial}{\partial t}(-\Delta\psi_{\varepsilon m}) \right\|_{L^2(0,T;(H^2(\Omega)\cap H_0^1(\Omega))')} \leqq C. \tag{3.41}$$

It follows from (3.30) that

$$\begin{aligned} a_0(\psi'_{\varepsilon m}, \phi) &= (F, \phi) - \mu\mathscr{A}(\psi_{\varepsilon m}, \phi) - \beta(\psi_{\varepsilon m}, \psi_{\varepsilon m}, \phi) \\ &\quad - g(\mathscr{J}'_\varepsilon(\psi_{\varepsilon m}), \phi) \end{aligned} \tag{3.42}$$

and it will suffice to show that each term in the right hand side of (3.42) is $\leqq \| \phi \|_{H^2(\Omega)} x$ (an $L^2(0,T)$ function).

This is obvious for the term (F, ϕ). We observe next that

$$\mu |\mathscr{A}(\psi_{\varepsilon m}, \phi)| \leqq \mu \, |\Delta\psi_{\varepsilon m}| \, |\Delta\phi|;$$

hence the result follows by virtue of (3.37). We then estimate

$$\begin{aligned} &\left| \int_\Omega \left(\frac{\partial}{\partial y}\psi_{\varepsilon m}\right)(\Delta\psi_{\varepsilon m}) \left(\frac{\partial}{\partial x}\phi\right) dx\,dy \right| \\ &\leqq |\Delta\psi_{\varepsilon m}| \left| \frac{\partial}{\partial y}\psi_{\varepsilon m} \right|_{L^4(\Omega)} \left| \frac{\partial\phi}{\partial x} \right|_{L^4(\Omega)} \\ &\leqq C|\Delta\psi_{\varepsilon m}|^2 \|\phi\|_{H^2(\Omega)} \leqq \text{(by virtue of (3.37))} \\ &\leqq C\|\phi\|_{H^2(\Omega)}, \end{aligned}$$

which gives the desired estimate for $\beta(\psi_{\varepsilon m}, \psi_{\varepsilon m}, \phi)$. Finally we observe that

$$\begin{aligned} |(\mathscr{J}'_\varepsilon(\psi), \phi)| &\leqq C \int_\Omega M(\psi)^{\varepsilon/2}[\phi_{xy}^2 + \tfrac{1}{4}(\phi_{xx} - \phi_{yy})^2]^{\frac{1}{2}} dx\,dy \\ &\leqq C\left(\int_\Omega M(\psi)\,dx\,dy\right)^{\frac{1}{2}} \|\phi\|_{H^2(\Omega)}. \end{aligned}$$

Hence the result follows, again using (3.37). This completes the proof of (3.41).

We now pass to the limit in m and in ε. By virtue of the preceding estimates, we can extract a subsequence, still denoted by $\psi_{\varepsilon m}$, such that

$$\begin{cases} \psi_{\varepsilon m} \to \psi \text{ in } L^\infty(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \text{ weak star}, \\ \Delta\psi_{\varepsilon m} \to \Delta\psi \text{ in } L^2(0,T; H_0^1(\Omega)) \text{ weakly}, \\ \dfrac{\partial}{\partial t}(-\Delta\psi_{\varepsilon m}) \to \dfrac{\partial}{\partial t}(-\Delta\psi) \text{ in } L^2(0,T; (H^2(\Omega) \cap H_0^1(\Omega))') \text{ weakly}. \end{cases} \tag{3.43}$$

It follows from (3.30) that

$$\begin{aligned} & a_0(\psi'_{\varepsilon m}, \phi - \psi_{\varepsilon m}) + \mu\mathscr{A}(\psi_{\varepsilon m}, \phi - \psi_{\varepsilon m}) + \beta\,(\psi_{\varepsilon m}, \psi_{\varepsilon m}, \phi) \\ & + g\mathscr{J}_\varepsilon(\phi) - g\mathscr{J}_\varepsilon(\psi_{\varepsilon m}) - (F, \phi - \psi_{\varepsilon m}) \\ & = g[\mathscr{J}_\varepsilon(\phi) - \mathscr{J}_\varepsilon(\psi_{\varepsilon m}) - (\mathscr{J}_\varepsilon(\psi_{\varepsilon m}), \phi - \psi_{\varepsilon m})] \geqq 0 \end{aligned} \tag{3.44}$$

(using the convexity of $\phi \to \mathscr{J}_\varepsilon(\phi)$).

Therefore,

$$\begin{aligned} & \int_0^T \left[\left(\frac{\partial}{\partial t}(-\Delta\psi_{\varepsilon m}), \phi\right) + \mu\mathscr{A}(\psi_{\varepsilon m}, \phi) + \beta(\psi_{\varepsilon m}, \psi_{\varepsilon m}, \phi) \right. \\ & \qquad \left. + g\mathscr{J}_\varepsilon(\phi) - (F, \phi - \psi_{\varepsilon m})\right] dt \\ & \geqq \int_0^T \left[a_0(\psi'_{\varepsilon m}, \psi_{\varepsilon m}) + \mu\mathscr{A}(\psi_{\varepsilon m}, \psi_{\varepsilon m}) + g\mathscr{J}_2(\psi_{\varepsilon m})\right] dt \\ & = \tfrac{1}{2} a_0(\psi_{\varepsilon m}(T), \psi_{\varepsilon m}(T)) + \mu \int_0^T \mathscr{A}(\psi_{\varepsilon m}, \psi_{\varepsilon m})\, dt \\ & \quad + g \int_0^T \mathscr{J}_\varepsilon(\psi_{\varepsilon m})\, dt. \end{aligned} \tag{3.45}$$

We use a compactness argument (as in Lions [23], Chap 1, Sec. 6.9) to prove that

$$\int_0^T \beta(\psi_{\varepsilon m}, \psi_{\varepsilon m}, \phi)\, dt \to \int_0^T \beta(\psi, \psi, \phi)\, dt;$$

the right hand side of (3.45) is lower semi-continuous, so that we obtain in the limit

$$\begin{aligned} & \int_0^T \left[\left(\frac{\partial}{\partial t}(-\Delta\psi)\right), \phi - \psi) + \mu\mathscr{A}(\psi, \phi - \psi) + \beta(\psi, \psi, \phi) + g\mathscr{J}(\phi) - g\mathscr{J}(\psi)\right] dt \\ & \geqq \int_0^T (F, \phi - \psi)\, dt. \end{aligned} \tag{3.46}$$

From this, one only obtains that ψ satisfies (3.23); this completes the proof of the "existence part" in Theorem 3.1.

PROOF OF THEOREM 3.1. *Uniqueness.*

Let ψ and ψ^* be two solutions of (3.12) (3.13) which satisfy (3.16) (3.17) (3.18). Taking $\phi = \psi^*$ (resp. $\phi = \psi$) in the inequality for ψ (resp ψ^*) and setting $\theta = \psi - \psi^*$ we obtain, after adding the results:

$$-a_0(\theta', \theta) - \mu\mathscr{A}(\theta, \theta) - \beta(\psi, \psi, \theta) + \beta(\psi^*, \psi^*, \theta) \geqq 0.$$

Hence

$$\tfrac{1}{2}\frac{d}{dt}\|\theta(t)\|_{H^1}^2 + \mu|\Delta\theta(t)|^2 \leqq -\beta(\psi,\theta,\theta)$$

and we deduce that $\theta = 0$ as in the proof of Theorem 6.10 of Lions [23], Chap. 1.

Proof of Theorem 3.2. We denote by ψ^μ the solutions obtained in Theorem 3.1. By virtue of the estimates (3.19), we can extract a subsequence, still denoted by ψ^μ, such that one has (3.20) (3.21) (3.22). We also notice that (3.37) implies that

(3.47) $\sqrt{\mu}\,\Delta\psi^\mu$ remains in a bounded set of $L^2(0,T;\, H_0^1(\Omega))$.

Since $-\Delta$ is an isomorphism from $H^2(\Omega) \cap H_0^1(\Omega)$ into $L^2(\Omega)$, it follows by transposition that it is also an isomorphism from $L^2(\Omega)$ into $(H^2(\Omega) \cap H^1(\Omega))'$ so that (3.22) is equivalent to

$$\frac{\partial\psi^\mu}{\partial t} \to \frac{\partial\psi}{\partial t} \quad \text{in } L^2(0,T;\, L^2(\Omega)) \text{ weakly.} \tag{3.48}$$

It follows from (3.12) that if we choose $\phi = \phi(t)$ to be a smooth function satisfying $\phi = 0$ on Γ, we have

$$\int_0^T \Big[a_0\Big(\frac{\partial\psi^\mu}{\partial t}, \phi - \psi^\mu\Big) + \mu\mathscr{A}(\psi^\mu, \phi - \psi^\mu) + \beta(\psi^\mu,\psi^\mu,\phi) + g\mathscr{J}(\phi) - g\mathscr{J}(\psi^\mu) - (F, \phi - \psi^\mu)\Big]\,dt \geqq 0. \tag{3.49}$$

By virtue of (3.47), $\int_0^T \mu\mathscr{A}(\psi^\mu, \phi - \psi^\mu)\,dt \to 0$ and by (3.21),

lower lim. $\displaystyle\int_0^T \mathscr{J}(\psi^\mu)dt \geqq \int_0^T \mathscr{J}(\psi)dt$. Using (3.21) (3.22)

we see that

$$\text{lower lim.} \int_0^T a_0\Big(\frac{\partial\psi^\mu}{\partial t},\psi^\mu\Big)dt \geqq \int_0^T a_0\Big(\frac{\partial\psi}{\partial t},\psi\Big)\,dt = \int_0^T \Big(\frac{\partial}{\partial t}(-\Delta\psi),\psi\Big)dt$$

so that (3.49) implies

$$\int_0^T \Big[a_0\Big(\frac{\partial\psi}{\partial t}, \phi - \psi\Big) + \beta(\psi,\psi,\phi) + g\mathscr{J}(\phi) - g\mathscr{J}(\psi) - (F, \phi - \psi)\Big]dt \geqq 0 \tag{3.50}$$

provided we check that

$$\int_0^T \beta(\psi^\mu,\psi^\mu,\phi)\,dt \to \int_0^T \beta(\psi,\psi,\phi)\,dt. \tag{3.51}$$

We have to verify that

$$\int_Q \psi_y^\mu \Delta\psi^\mu \phi_x\,dQ \to \int_Q \psi_y \Delta\psi\,\phi_x\,dQ. \tag{3.52}$$

We can take ϕ as smooth as we please (by a density argument), so that

$\Delta\phi^\mu\ \phi_x \to \Delta\psi\ \phi_x$ in, say, $L^2(Q)$ weakly. By (3.21) and (3.48), we have

$$\psi_y^\mu \to \psi_y \quad \text{in } L^2(0,T;\, H^1(\Omega)) \text{ weakly,}$$

$$\frac{\partial\psi_y^\mu}{\partial t} \to \frac{\partial\psi_y}{\partial t} \quad \text{in } L^2(0,T;\, H^1(\Omega)) \text{ weakly.}$$

Hence, by using a compactness result (cf. Lions [24]), it follows that

$$\psi_y^\mu \to \psi_y \text{ in } L^2(Q) \text{ strongly}$$

so that (3.52) and (3.51) follow. Thus (3.50) is proved and (3.23) follows by a standard argument.

REMARK 3.3. The uniqueness in Theorem 3.2 is an open problem, for $g > 0$. In case $g = 0$, uniqueness is known. Cf. Yudovich [40], Lions [23].

REMARK 3.4. It would be interesting to extend to the present situation results obtained for the case $g = 0$ by D. G. Ebin and J. Marsden [9], T. Kato [17] and H. S. G. Swann [38]. We believe the results to be valid for the case $g > 0$ but the "j-terms" lead to serious technical difficulties.

4. Heat transfer in a Bingham fluid

Let us consider now, following Duvaut and Lions [8], a situation where the viscosity μ depends on the temperature θ of the fluid.

Let $\lambda \to \mu(\lambda)$ be a continuous function defined on ω, satisfying

$$0 < \mu_0 \leqq \mu(\lambda) \leqq \mu_1 < \infty \qquad \forall\lambda \in \mathbb{R}. \tag{4.1}$$

We set now

$$a(\theta;\, u, v) = 2\int_\Omega \mu(\theta)\, D_{ij}(u)\, D_{ij}(v)\, dx^\dagger \tag{4.2}$$

and we introduce

$$F(\theta, u) = 2(2\mu(\theta)\, D_{II}(u) + g(D_{II}(u))^{\frac{1}{2}}). \tag{4.3}$$

Then the speed of the flow u and the temperature θ of the fluid are shown to satisfy to

† In case $\mu(\theta) = \mu$, $a(\theta;\, u, v) = \mu\, a(u, v)$ with the notations of Sections 2 and 3.

$$(4.4) \quad \left[\begin{array}{l}\left(\dfrac{\partial u}{\partial t}, v-u\right)+a(\theta; v-u)+b(u,u,v-u)+gj(v)-gj(v) \geqq (f, v-u) \\ \forall v \in V,\end{array}\right.$$

$$(4.5) \qquad \frac{\partial \theta}{\partial t} - k\Delta\theta = c\, F(\theta, u) + \tilde{f}, \qquad k, c > 0,$$

where $\tilde{f}$ is given in $\Omega \times]0, T[$, with the boundary conditions

$$(4.6) \qquad u = 0 \text{ on } \Gamma^{\dagger}$$

$$(4.7) \qquad k\frac{\partial \theta}{\partial n} + q\theta = 0 \text{ on } \Gamma, \qquad q \geqq 0$$

and with the initial conditions

$$(4.8) \qquad u(x,0) = u_0(x),\ \theta(x,0) = \theta_0(x), (u_0, \theta_0 \text{ given}).$$

The following theorem is proved in [8]:

THEOREM 4.1. *We assume that the space dimension* $n = 2$, *and that*

$$(4.9) \qquad f \in L^2(0,T; V'),\ \tilde{f} \in L^1(Q),\ Q = \Omega \times]0,T[,$$

$$(4.10) \qquad u_0 \in H,\ \theta_0 \in L^1(\Omega).$$

Then there exists u, θ, *a solution of* (4.5) $\cdots$ (4.8) *such that*

$$(4.11) \qquad u \in L^2(0,T; V),\ \frac{\partial u}{\partial t} \in L^2(0,T; V'),$$

$$(4.12) \qquad \theta \in L^{\beta}(Q),\ \textit{where } 1 < \beta < 3/2.$$

REMARK 4.1. The solution θ in $L^{\beta}(Q)$ of (4.5) (4.7) (4.8) is meant in a weak sense, after integration by parts.

REMARK 4.2. The *uniqueness* in Theorem 4.1 is an open question. If $n = 3$, the existence appears also to be an open question.

5. Remarks and problems

5.1 *The case* $g \to +\infty$. Let u_g be the solution given by Theorem 2.1, where μ is fixed > 0. Let us suppose that

$$(5.1) \qquad u_0 = 0$$

and that

$$(5.2) \qquad f \in L^{\infty}(0,T; H).$$

---

† This condition is satisfied when $u(t) \in V$.

Then, using in particular, an inequality of L. Nirenberg (private communication) extended by M. J. Strauss [37], one shows (cf. [6], Chap. 6, Sec. 5) that

(5.3) $$u_g = 0 \text{ for } g \geqq g_c.$$

We call "rigid domain" the region of the flow where

(5.4) $$D_{ij}(u_g) = 0 \qquad \forall\, i,j.$$

It seems natural that this rigid domain increases with g; this conjecture, made in [6], is confirmed by numerical experiments (cf. M. Fortin [12], D. Bégis [2]).

5.2. *Multipliers.* One can state the variational inequalities met in this paper in terms of *equalities* using *Lagrange multipliers* (not uniquely defined).

We refer to Duvaut and Lions [6] [8] for these equalities.

5.3. *Regularity.* In problems of variational inequalities, there is, in general, a "regularity yield" (counter example of Shamir [34], Brézis and Stampacchia [5] and Brézis [4]). Is it possible to prove regularity theorems analogous to those of Brézis [4] for the variational inequality "without b term", i.e.,

(5.5) $$\left(\frac{\partial u}{\partial t}, v-u\right) + \mu a(u,v-u) + gj(v) - gj(u) \geqq (f,v-u) \qquad \forall\, v \in V?$$

It is very likely that one cannot extend all results known in the case $g = 0$, such as those of Serrin [33].

5.4. Let us suppose that the right hand side f and the initial data u_0 are *random* functions in Theorem 2.2. (i.e., when space dimension equals 3). Then R. Temam [39] has proved that one can find a *measurable* family of solutions. A general problem which arises in this context is then: *Is it possible to estimate the probability of having uniqueness*?

5.5. For the extension of the theory of C. Foias and G. Prodi [10], [11] (which is established for the case $g=0$) to the case $g > 0$, we refer to Pop Cioranescu [31].

5.6. It would be interesting to extend to the Bingham flows the results of *stability* known for the Navier Stokes equations (cf. G. Iooss [15], O. A. Ladyzenskaya [19] and D. Sattinger [32]).

5.7. One can extend to the case $g > 0$ some of the results of Simonenko [35] [36] relative to the case "$g = 0$".

5.8. For the study of almost periodic solutions of Bingham's variational inequalities, we refer to Biroli [3] where some extensions of results of Amerio-Prouse [1] are given.

5.9. One meets (G. Duvaut, private communication) *free boundary* problems for Bingham's fluids; they seem to lead to open problems.

5.10. For Bingham's flows in *non-cylindrical domains*, we refer to B. Margolis [28] who gives extensions of results for Navier Stokes equations in non-cylindrical domains obtained in H. Fujita and N. Sauer [13], Lions [25], H. Morimoto [30] (where one studies the existence of *periodical* solutions).

5.11. Other models, introduced in the case $g = 0$ by Ladyzenskaya [20], S. Kaniel [16], Lions [26], are solved for the case $g > 0$ in B. Margolis [29].

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